

ON MULTILINEAR OSCILLATORY INTEGRALS, NONSINGULAR AND SINGULAR

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ABSTRACT. Basic questions concerning nonsingular multilinear operators with oscillatory factors are posed and partially answered. L^p norm inequalities are established for multilinear integral operators of Calderón-Zygmund type which incorporate oscillatory factors e^{iP} , where P is a real-valued polynomial. A related problem concerning upper bounds for measures of sublevel sets is solved.

1. INTRODUCTION

Consider multilinear functionals

$$(1.1) \quad \Lambda_\lambda(f_1, f_2, \dots, f_n) = \int_{\mathbb{R}^m} e^{i\lambda P(x)} \prod_{j=1}^n f_j(\pi_j(x)) \eta(x) dx$$

where $\lambda \in \mathbb{R}$ is a parameter, $P : \mathbb{R}^m \rightarrow \mathbb{R}$ is a measurable real-valued function, $m \geq 2$, and $\eta \in C_0^1(\mathbb{R}^m)$ is compactly supported. Each π_j denotes the orthogonal projection from \mathbb{R}^m to a linear subspace $V_j \subset \mathbb{R}^m$ of any dimension $\kappa \leq m-1$, and $f_j : V_j \rightarrow \mathbb{C}$ is always assumed to be locally integrable with respect to Lebesgue measure on V_j . We assume for simplicity that κ is independent of j .

The integral Λ_λ is well-defined if all f_j belong to L^∞ , and satisfies $|\Lambda_\lambda(f_1, \dots, f_n)| \leq C \prod_j \|f_j\|_{L^\infty}$.

Definition 1.1. A measurable real-valued function P is said to have the power decay property, relative to a collection of subspaces $\{V_j\}$ of \mathbb{R}^m , on an open set¹ $U \subset \mathbb{R}^m$ if for any $\eta \in C_0^1(U)$ there exist $\varepsilon > 0$ and $C < \infty$ such that

$$(1.2) \quad |\Lambda_\lambda(f_1, \dots, f_n)| \leq C(1 + |\lambda|)^{-\varepsilon} \prod_{j=1}^n \|f_j\|_{L^\infty} \quad \text{for all } f_j \in L^\infty(V_j) \text{ and all } \lambda \in \mathbb{R}.$$

Moreover, for η supported in any fixed compact set, ε may be taken to be independent of η , and $C = O(\|\eta\|_{C^1})$.

The goal of this paper is to characterize those data $(P, \{V_j\})$ for which the power decay property holds. A necessary condition is that P can not be expressed as a linear combination of measurable functions $p_j \circ \pi_j$. Indeed, if $P = \sum_j p_j \circ \pi_j$ then for $f_j = e^{-i\lambda p_j} \in L^\infty$ one has $e^{i\lambda P(x)} \prod_j f_j(\pi_j(x)) \equiv 1$, and there is consequently no decay.

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¹Our theorems concern only polynomial phases P , for which we will show that nondegeneracy is independent of U .

Definition 1.2. A polynomial P is said to be degenerate (relative to $\{V_j\}$) if there exist polynomials $p_j : V_j \rightarrow \mathbb{R}$ such that $P = \sum_{j=1}^n p_j \circ \pi_j$. Otherwise P is nondegenerate.

In the case $n = 0$, where the collection of subspaces $\{V_j\}$ is empty, P is considered to be nondegenerate if and only if it is nonconstant.

We will show in Lemma 3.4 that degeneracy is also equivalent to the existence, in some nonempty open set, of a representation of the more general form $P = \sum_{j=1}^n h_j \circ \pi_j$ where the h_j are arbitrary distributions rather than polynomials.

More generally, a (measurable) function P is said to be degenerate if it can be expressed as $\sum_j p_j \circ \pi_j$ for some (measurable) functions p_j , or distributions. But we will restrict attention to polynomials henceforth.

Question. Is the power decay property equivalent to nondegeneracy, for real-valued polynomials?

We were led to this question by a problem concerning multilinear singular integral operators; see Theorems 2.6 and 2.7 below. We have been able answer it, always in the affirmative, only in special cases.

Numerous variants can be formulated. One can ask when (1.2) holds with the right-hand side replaced by $\Theta(\lambda) \prod_{j=1}^n \|f_j\|_{L^\infty}$ where $\Theta(\lambda)$ is a specific function of $(1 + |\lambda|)$, or whether there merely exists some function Θ tending to zero as $|\lambda| \rightarrow \infty$ for which it holds. When (1.2) does hold, one can ask what is the optimal power ε in (1.2). We will focus on the first formulation, which is the one most directly relevant to our applications to multilinear singular integral operators, and is possibly the most fundamental.

The extreme formulation in which all f_j are measured in the strongest Lebesgue norm L^∞ is the essence of the matter. Since $|\Lambda_\lambda(f_1, \dots, f_n)| \leq C \|f_k\|_{L^1} \prod_{j \neq k} \|f_j\|_{L^\infty}$ uniformly in λ for any k , a simple interpolation argument shows that if (1.2) does hold, then a decay estimate of the same type holds with $\prod_j \|f_j\|_\infty$ replaced by $\prod_j \|f_j\|_{p_j}$ for various n -tuples of indices $p_j \in (1, \infty]$. More precisely, if the integral converges absolutely whenever each $f_j \in L^{p_j}$, then $|\Lambda_\lambda(f_1, \dots, f_n)| \leq C(1 + |\lambda|)^{-\varepsilon} \prod_j \|f_j\|_{q_j}$ whenever each $q_j > p_j$, where $\varepsilon > 0$ depends on $\{p_j\}, \{q_j\}$, and other n -tuples q can in general be obtained via further interpolations. On the other hand, the combinations of exponents p_j for which the integral is guaranteed to converge absolutely depend strongly on $\{V_j\}$.

When $n = 0$, that is, when $\{V_j\}$ is empty, one is dealing with oscillatory integrals $\Lambda_\lambda = \int e^{i\lambda P(x)} \eta(x) dx$, which are of the first type in the terminology of Stein [10]. The general case can also be regarded as concerning oscillatory integrals of the first type. Indeed, (1.2) with L^∞ bounds on the functions f_j is equivalent to $|\int e^{i\lambda \phi(x)} \eta(x) dx| \leq C|\lambda|^{-\varepsilon}$ uniformly for all phase functions of the form $\phi = P - \sum_j h_j \circ \pi_j$, where the h_j are arbitrary real-valued measurable functions; one implication is tautologous, while the second is nearly trivial and is proved in [1].

The regularity condition $\eta \in C^1$ is rather arbitrary. If η is merely Hölder continuous then for any $s < \infty$, η may be decomposed as a smooth function whose C^s norm is $O(|\lambda|^{C\delta})$ plus a remainder which is $O(|\lambda|^{-\delta})$ in supremum norm. If (1.2) holds for all $\eta \in C_0^s$ with a constant C which is $O(\|\eta\|_{C^s})$ then it follows from this decomposition, with $\delta = \varepsilon/2C$, that it continues to hold for all Hölder continuous η .

$\prod_{j=1}^n f_j \circ \pi_j$ could be replaced by $\prod_{j=1}^n g_j \circ \ell_j$ in (1.1), where each $\ell : \mathbb{R}^m \rightarrow \mathbb{R}^\kappa$ is linear and has full rank, without any increase in generality. For any function $g \circ \ell$ may be expressed as $f \circ \pi$, where π is the orthogonal projection of \mathbb{R}^m onto the orthocomplement V of the nullspace of ℓ ; $f \in L^p(V)$ if and only if $g \in L^p(\mathbb{R}^\kappa)$ with comparable norms.

The linear prototype for power decay is the inequality (see Stein [10], p. 416 and Phong and Stein [7])

$$(1.3) \quad \left| \int e^{i\lambda P(x,y)} f(x)g(y)\eta(x,y) dx dy \right| = O(|\lambda|^{-\varepsilon} \|f\|_{L^2} \|g\|_{L^2}),$$

with suitable uniformity, whenever P is a real-valued polynomial, of bounded degree, which is nondegenerate in the sense that some mixed partial derivative $\partial^{\alpha+\beta} P / \partial x^\alpha \partial y^\beta$, with both $\alpha, \beta \neq 0$, does not vanish identically; this is equivalent to the impossibility of decomposing $P(x, y)$ as $p(x) + q(y)$.

A related result is the lemma of van der Corput, which asserts for instance that $\int_a^b e^{i\lambda\varphi(t)} dt = O(1 + |\lambda|)^{-\varepsilon}$ provided that φ is real-valued and that some derivative of φ is bounded away from zero (in the case of the first derivative, it is also assumed to be monotone). Here the upper bound for the integral depends only on a lower bound for the derivative; no upper bound on φ or its derivatives, hence in some sense no *a priori* smoothness condition, are imposed. Higher-dimensional versions of van der Corput's lemma, likewise without assumptions of upper bounds or extra smoothness, were established in [1].

Our nondegeneracy condition replaces the hypothesis of a nonvanishing derivative. (1.2), like (1.3), is invariant under the replacement of P by $P - \sum_j p_j \circ \pi_j$, for arbitrary real-valued measurable functions p_j , and there is no *a priori* smoothness condition or upper bound on $P - \sum_j p_j \circ \pi_j$. Formally the power decay property more closely resembles (1.3), but our analysis is closely related to [1], and combinatorial issues lurk here as they did there.

2. RESULTS

2.1. Further definitions. Given a degree d , fix a norm $\|\cdot\|_{\mathcal{P}_d}$ on the finite-dimensional vector space \mathcal{P}_d of all polynomials in \mathbb{R}^m of degree $\leq d$. Given d , the norm of P relative to $\{V_j\}$ is defined to be $\inf \|P - \sum_j p_j \circ \pi_j\|_{\mathcal{P}_d}$, where the infimum is taken over all real-valued polynomials p_j of degrees $\leq d$. Polynomials P_α are said to be uniformly nondegenerate if they are all of degrees $\leq d$ for some finite d , and if there exists $c > 0$ such that the norm of P_α relative to $\{V_j\}$ (and relative to this degree d) is $\geq c$ for all α .

More generally, if we are given a collection of subspaces $\{V_j^\alpha : \alpha \in A\}$ and a polynomial P_α of uniformly bounded degree, we may still define uniform nondegeneracy by requiring that $\inf_\alpha \inf_{\{p_j\}} \|P_\alpha - \sum_j p_j \circ \pi_j^\alpha\|_{\mathcal{P}_d} \geq c > 0$, where π_j^α is the orthogonal projection onto V_j^α .

These definitions are independent of d provided only that each P has degree $\leq d$, a simple consequence of the equivalence of any two norms on a finite-dimensional vector space, and the fact that if $P = \sum_j p_j \circ \pi_j$ then $P = \sum_j \tilde{p}_j \circ \pi_j$ where \tilde{p}_j is the sum of all terms of degrees $\leq \text{degree}(P)$ in the decomposition of π_j as a linear combination of monomials.

Because the family of polynomials of degree $\leq d$ has finite dimension, it is easily verified that the infimum defining the relative norm is actually assumed for some polynomials p_j . Thus either P is degenerate, or the infimum is strictly positive.

Definition 2.1. A collection of subspaces $\{V_j\}$ is said to have the power decay property if every real-valued polynomial P which is nondegenerate relative to $\{V_j\}$ has the power decay property (1.2), in every open set U .

A collection of subspaces $\{V_j\}$ is said to have the uniform power decay property if (1.2) holds, with uniform constants C, ε , for any family of real-valued polynomials of bounded degrees which are uniformly nondegenerate relative to $\{V_j\}$.

A related concept turns out to be somewhat easier to analyze. To any distribution $h \in \mathcal{D}'(V_j)$ is naturally associated a distribution $h \circ \pi_j \in \mathcal{D}'(\mathbb{R}^m)$. Denote by V_j^\perp the orthogonal complement of V_j . Then $h \circ \pi_j$ is annihilated by any first-order constant-coefficient differential operators of the form $w \cdot \nabla$, where $w \in V_j^\perp$.

Definition 2.2. A polynomial P is said to be simply nondegenerate if there exists a differential operator L of the form $L = \prod_{j=1}^r (w_j \circ \nabla)$, with each $w_j \in V_j^\perp$, such that $L(P)$ does not vanish identically.

Simple nondegeneracy implies nondegeneracy. Indeed, for any distribution f defined on some V_j , the j -th factor of L annihilates $f \circ \pi_j$, and hence L does so since all factors commute. Thus $L(P - \sum_j f_j \circ \pi_j) \equiv L(P)$.

The converse is not true in general. Consider the case $\kappa = 1$ where each subspace V_j has dimension one. Let \mathcal{L} be any homogeneous constant-coefficient linear partial differential operator with real coefficients, and let σ be its symbol. Identify the dual space of \mathbb{R}^m with \mathbb{R}^m via the inner product. Let e be any unit vector and let $\pi(x) = \langle x, e \rangle e$. Then \mathcal{L} annihilates every function $f \circ \pi$ if and only if $\sigma(e) = 0$. To create examples of nondegenerate polynomials, fix any such nonelliptic \mathcal{L} , choose $\{e_j\}$ to be any finite collection of distinct unit vectors satisfying $\sigma(e_j) = 0$, and choose any real-valued polynomial P such that $\mathcal{L}(P)$ does not vanish identically. This forces P to be nondegenerate relative to $\{V_j\}$, where V_j is the span of e_j . If $m \geq 3$ then $P, \mathcal{L}, \{V_j\}$ can be chosen so that P has degree two, yet $\{V_j\}$ has arbitrarily large finite cardinality.

2.2. Decay for nonsingular oscillatory multilinear functionals. We now state several theorems asserting that nondegeneracy implies the power decay property, under various auxiliary hypotheses. To formulate the first of these, let $\{V_j : 1 \leq j \leq n\}$ be a collection of subspaces of \mathbb{R}^m of dimension κ . We say that they are in general position if any subcollection of cardinality $k \geq 1$ spans a subspace of dimension $\min(k\kappa, m)$. It is elementary that for subspaces in general position, if each f_j belongs to $L^2(\mathbb{R})$, then the product of any k functions $f_j \circ \pi_j$ belongs to $L^2(\mathbb{R}^m)$ provided that $k\kappa \leq m$. When $k\kappa \leq 2m$ the product belongs to L^1 , by Cauchy-Schwarz. Therefore the integral defining Λ_λ converges absolutely, and $|\Lambda_\lambda(f_1, \dots, f_n)| \leq C \prod_j \|f_j\|_{L^2}$, uniformly in $\lambda \in \mathbb{R}$.

Theorem 2.1. *Suppose that $n < 2m$. Then any family $\{V_j : 1 \leq j \leq n\}$ of one-dimensional subspaces of \mathbb{R}^m which lie in general position has the uniform power decay property. Moreover under these hypotheses*

$$(2.1) \quad |\Lambda_\lambda(f_1, \dots, f_n)| \leq C(1 + |\lambda|)^{-\varepsilon} \prod_{j=1}^n \|f_j\|_{L^2}$$

for all polynomials P of bounded degree which are uniformly nondegenerate with respect to $\{V_j\}$, for all functions $f_j \in L^2(\mathbb{R}^1)$, with uniform constants $C, \varepsilon \in \mathbb{R}^+$.

The case $n = m$ is known; Phong, Stein, and Sturm [8] have obtained much more precise results on the exponent in the power decay estimate, phrased in terms of the reduced Newton polyhedron of P . The case $n < m$ follows from $n = m$. Indeed, choose coordinates $(x', x'') \in \mathbb{R}^n \times \mathbb{R}^{m-n}$, in such a way that the first factor \mathbb{R}^n equals the span of the subspaces V_j . If $\partial P / \partial x''$ does not vanish identically then (1.2) does hold, as one sees by integrating with respect to x'' for generic fixed x' and using well-known results for oscillatory integrals of the first type in the sense of [10]. If P is independent of x'' then matters have been reduced to the case where the ambient dimension is n .

As will be clear from its proof, (2.1) continues to hold with uniform constants C, ε if the collection $\{V_j\}$ varies over a compact subset of the open subset of the relevant Grassmannian manifold consisting of all such collections in general position, and the polynomials P are uniformly nondegenerate relative to $\{V_j\}$.

The condition $n < 2m$ is necessary for (2.1), with $\|f_j\|_{L^2}$ on the right-hand side rather than $\|f_j\|_{L^\infty}$ as in Definition 1.1. To see this in the main case where $\{V_j\}$ span \mathbb{R}^m , consider without loss of generality the case $n = 2m$, and let each f_j equal the characteristic function of $[-\delta, \delta]$. If δ is chosen to be a small fixed constant times $|\lambda|^{-1}$, then $|\Lambda_\lambda(f_1, \dots, f_n)| \geq c\delta^m$. On the other hand $\prod_j \|f_j\|_{L^2} = c'\delta^{n/2}$. It follows from the same construction that if $n > 2m$ then the integral defining $\Lambda_\lambda(f_1, \dots, f_n)$ will in general not even be absolutely convergent for general $f_j \in L^2$.

Theorem 2.2. *Any collection of subspaces $\{V_j\}$ of codimension one has the uniform power decay property.*

In particular, when $m = 2$ then $\kappa = 1 = m - 1$ and hence nondegeneracy of P implies (1.2), no matter how large the collection of subspaces V_j may be.

Theorem 2.3. *Let m, κ be arbitrary. Then any simply nondegenerate polynomial has the power decay property (1.2) in every open set.*

More precisely, let $d \in \mathbb{N}$ and $c > 0$. Let $L = \prod_{j=1}^n (w_j \cdot \nabla)$ where each $w_j \in V_j^\perp$ is a unit vector. Then there exist $C < \infty$ and $\varepsilon > 0$ such that for any real-valued polynomial P of degree $\leq d$ such that $\max_{|x| \leq 1} |L(P)(x)| \geq c$,

$$|\Lambda_\lambda(f_1, \dots, f_n)| \leq C(1 + |\lambda|)^{-\varepsilon} \prod_j \|f_j\|_\infty$$

for all functions $f_j \in L^\infty$ and all $\lambda \in \mathbb{R}$.

In each of these theorems, the regularity hypothesis on η can be relaxed to Hölder continuity, as discussed above.

Theorem 2.2 directly implies Theorem 2.3. Indeed, if w_j are as in the latter theorem, define $\tilde{V}_j = w_j^\perp$, thus obtaining codimension one subspaces $\tilde{V}_j \supset V_j$. Each $f_j \circ \pi_j$ can be rewritten as $\tilde{f}_j \circ \tilde{\pi}_j$ where $\tilde{\pi}_j$ is the orthogonal projection onto \tilde{V}_j and $\tilde{f}_j(y) = f_j \circ \pi_j(y)$ for $y \in \tilde{V}_j$. The hypothesis $L(P) \neq 0$ guarantees that P is nondegenerate relative to $\{\tilde{V}_j\}$, with appropriate uniformity. On the other hand, we will show in Proposition 3.1 that for $\kappa = m - 1$, nondegeneracy is equivalent to simple nondegeneracy, so that Theorem 2.2 conversely implies Theorem 2.3.

The next two theorems are the best we have been able to do for general values of the dimension κ .

Theorem 2.4. *Let $\kappa \in \{1, 2, \dots, m - 1\}$. Let $\{V_j : 1 \leq j \leq n\}$ be a collection of κ -dimensional subspaces of \mathbb{R}^m , lying in general position. If*

$$n < \frac{2m}{\kappa}$$

then $\{V_j\}$ has the uniform decay property, in its L^2 formulation (2.1), with uniform bounds if P belongs to a family of uniformly nondegenerate polynomials.

For $\kappa = m - 1$ we have already shown that the conclusion holds with no restriction on n .

When the codimension $m - \kappa$ is relatively small but strictly greater than one then the next result is superior.

Theorem 2.5. *Let $\{V_j : 1 \leq j \leq n\}$ be a finite collection of subspaces of \mathbb{R}^m , each having dimension κ and lying in general position. If*

$$n \leq \frac{m}{m - \kappa}$$

then $\{V_j\}$ has the uniform decay property.

2.3. Singular integrals. For any real valued polynomial $P(x, t)$ of degree d , consider the operator

$$(2.2) \quad T(f, g)(x) = \int_{-\infty}^{\infty} e^{iP(x, t)} f(x - t) g(x + t) t^{-1} dt$$

where it is initially assumed that $f, g \in C_0^1$, the class of all continuously differentiable functions having compact supports, and the integral is taken in the principal-value sense. One of the purposes of this note is to establish the following L^p bounds for these operators.

Theorem 2.6. *For any exponents $p_1, p_2, q \in (0, \infty)$ such that $q^{-1} = p_1^{-1} + p_2^{-1}$, $p_1, p_2 > 1$ and $q > 2/3$, and any degree $d \geq 1$, there exists $C < \infty$ such that $\|T(f, g)\|_q \leq C \|f\|_{p_1} \|g\|_{p_2}$ for all $f, g \in C_0^1$, uniformly for all real-valued polynomials P of degrees $\leq d$.*

This answers a question raised by Lacey and Thiele. The cases $d = 0, 1, 2$ were previously known. Indeed, the case $d = 0$ is a celebrated theorem of Lacey and Thiele [4]. The case $d = 1$, that is $P(x, t) = a_0x + a_1t$, can be reduced to $d = 0$ by replacing f by $\tilde{f}(x) = e^{ia_0x/2 - ia_1x/2} f(x)$ and g by $\tilde{g}(x) = e^{ia_0x/2 + ia_1x/2} f(x)$. The case $d = 2$, that is, $P(x, t) = a_0x^2 + a_1xt + a_2t^2 + b_1x + b_2t$, is likewise reducible to $d = 0$ by the substitution $\tilde{f}(x) = e^{-ia_1x^2/4 + ia_2x^2/2} f$, $\tilde{g}(x) = e^{ia_1x^2/4 + ia_2x^2/2} g$. But for $d \geq 3$ no such simple reduction exists.

Theorem 2.6 is a bilinear analogue of a theorem of Ricci and Stein [9], who proved L^p estimates for linear operators $f \mapsto \int e^{iP(x, t)} f(x - t) K(t) dt$, for arbitrary real-valued polynomials P and Calderón-Zygmund kernels K . It is a special case of the following more general result. Let $n \geq 1$, $\Gamma = \{\xi \in \mathbb{R}^{n+1} : \xi_1 + \xi_2 + \cdots + \xi_{n+1} = 0\}$, and Γ' be the orthogonal complement of a subspace of Γ such that the dimension of $\Gamma' \cap \Gamma$ is k . Let K be a k -dimensional Calderón-Zygmund kernel on $\Gamma' \cap \Gamma$, that is, K is Lipschitz continuous except at the origin, $K(r\gamma) \equiv r^{-k} K(\gamma)$ for all $r > 0$ and $\gamma \neq 0$, and $\int_{S^{k-1}} K d\sigma = 0$ where σ denotes surface measure on the unit sphere S^{k-1} in $\Gamma' \cap \Gamma$. We define the n -linear operator T by

$$(2.3) \quad T(f_1, f_2, \dots, f_n)(x) = \int_{\Gamma' \cap \Gamma} f_1(x + \gamma_1) f_2(x + \gamma_2) \cdots f_n(x + \gamma_n) K(\gamma) d\gamma,$$

for $x \in \mathbb{R}^1$, where $d\gamma$ is Lebesgue measure on $\Gamma' \cap \Gamma$, and $\gamma_i \in \mathbb{R}^1$ is the i -th coordinate of $\gamma \in \mathbb{R} \times \mathbb{R} \cdots \times \mathbb{R}$ as an element of \mathbb{R}^{n+1} . The integral is interpreted in the principal-value sense, so that $T(f_1, \dots, f_n)(x)$ is well-defined provided that each $f_j \in C^1$ has compact support. We always assume that any $k + 1$ variables in $\{x + \gamma_1, \dots, x + \gamma_n, x\}$ are linearly independent.

To such an operator T and to any real-valued polynomial $P(x, \gamma_1, \dots, \gamma_n)$ we associate the multilinear operator

$$(2.4) \quad T_P(f_1, f_2, \dots, f_n)(x) = \int_{\Gamma' \cap \Gamma} e^{iP(x, \gamma_1, \dots, \gamma_n)} f_1(x + \gamma_1) f_2(x + \gamma_2) \cdots f_n(x + \gamma_n) K(\gamma) d\gamma,$$

which is again well-defined when each $f_j \in C_0^1$.

Write $p_0 = q'$.

Theorem 2.7. *Suppose that $n \leq 2k$. Then for any real-valued polynomial P , T_P maps $\otimes_{j=1}^n L^{p_j}$ boundedly to L_0^p whenever $p_0 > n^{-1}$, $1 < p_j \leq \infty$ and $p_0^{-1} = \sum_j p_j^{-1}$, provided that*

$$(2.5) \quad \frac{1}{p_{i_1}} + \frac{1}{p_{i_2}} + \cdots + \frac{1}{p_{i_r}} < \frac{2k + r + 1 - n}{2}$$

for all $0 \leq i_1 < i_2 < \cdots < i_r \leq n$ and all $1 \leq r \leq n + 1$. This conclusion holds uniformly for all polynomials of degrees $\leq D$, for any $D < \infty$.

Under these hypotheses, the nonoscillatory case $P \equiv 0$ was treated in [6]. Theorems 2.6 and 2.7 will be proved by combining previously known results for nonoscillatory multilinear singular integral operators with our new results for nonsingular oscillatory integrals.

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2.4. Bounds for sublevel sets. We are able to prove a weaker consequence of the decay property in full generality, providing some evidence that nondegeneracy might always imply it. For any Lebesgue measurable functions g_j which are finite almost everywhere, for any $\varepsilon > 0$, and for any bounded subset $B \subset \mathbb{R}^m$ consider the sublevel sets

$$(2.6) \quad E_\varepsilon = \{x \in B : |P(x) - \sum_j g_j(\pi_j(x))| < \varepsilon\}.$$

Theorem 2.8. *Suppose that a real-valued polynomial P is nondegenerate with respect to a finite collection of subspaces $\{V_j\}$ of \mathbb{R}^m . Then there exists $\delta > 0$ so that for any bounded subset B of \mathbb{R}^m there exists $C < \infty$ such that for any measurable functions g_j defined on V_j , for any $\varepsilon > 0$, the associated sublevel sets satisfy*

$$(2.7) \quad |E_\varepsilon| \leq C\varepsilon^\delta.$$

If a real-valued measurable function P has the power decay property (1.2), then the sublevel set bound (2.7) holds. Indeed, fix a cutoff function $0 \leq h \in C_0^\infty(\mathbb{R})$ satisfying $h(t) = 1$ whenever $|t| \leq 1$. Fix also $\zeta \in C_0^\infty(\mathbb{R}^m)$ such that $\zeta \equiv 1$ on B . Then

$$\begin{aligned} |\{x \in B : |P(x) - \sum_j g_j(\pi_j(x))| < \varepsilon\}| &\leq \int h[\varepsilon^{-1}(P - \sum_j g_j \circ \pi_j)(x)] \zeta(x) dx \\ &= (2\pi)^{-1} \varepsilon \int_{\mathbb{R}} \widehat{h}(\varepsilon\lambda) \int e^{i\lambda(P(x) - \sum_j g_j(\pi_j(x)))} \zeta(x) dx d\lambda. \end{aligned}$$

The inner integral is $O(1 + |\lambda|)^{-r}$ for some $r \in (0, 1)$ by the power decay property applied to $f_j = e^{i\lambda g_j}$. Inserting this into the double integral gives a majorization

$$|\{x \in B : |P(x) - \sum_j g_j(\pi_j(x))| < \varepsilon\}| \leq C \int_{\mathbb{R}} \varepsilon(1 + \varepsilon|\lambda|)^{-2}(1 + |\lambda|)^{-r} d\lambda \leq C\varepsilon^{-r}.$$

The remainder of the paper is devoted to proofs of the theorems. Certain algebraic aspects of nondegeneracy come into play in the analysis and are developed in §3; this material is used throughout the paper. The simply nondegenerate case is treated in §4. The analysis when the dimension κ equals 1 involves an additional ingredient and is presented in §5. Applications to multilinear operators with Calderón-Zygmund singularities and oscillatory factors are carried out in §6. The bounds for measures of sublevel sets are established in §7. Finally, Theorems 2.4 and 2.5 are proved in the final section.

3. ALGEBRAIC ASPECTS OF NONDEGENERACY

This section develops various general properties, largely algebraic in nature, of nondegeneracy.

As noted above, simple nondegeneracy implies nondegeneracy. In the codimension one case they are equivalent:

Proposition 3.1. *If each subspace V_j has codimension one, that is if $\kappa = m - 1$, then any nondegenerate polynomial is simply nondegenerate.*

Proof. Let a polynomial P and distinct subspaces V_j of codimension 1 be given. Let w_j be unit vectors orthogonal to V_j , let $L = \prod_{j=1}^n (w_j \cdot \nabla)$, and let $L' = \prod_{j=1}^{n-1} (w_j \cdot \nabla)$. We must show that if $LP \equiv 0$, then P is degenerate.

Proceed by induction on n , which is the total number of subspaces V_j . Then since $L'(w_n \cdot \nabla)P \equiv 0$, $(w_n \cdot \nabla)P = \sum_{j=1}^{n-1} q_j \circ \pi_j$ for some polynomials q_j . By a rotation of the coordinate system it may be arranged that $w_n \cdot \nabla = \partial_{x_m}$.

$(w_n \cdot \nabla)(f \circ \pi_j) = (D_j f) \circ \pi_j$ for some nonvanishing constant-coefficient real vector fields D_j , for w_n cannot be orthogonal to V_j for $j < n$ since it is orthogonal to V_n , and these codimension one subspaces are distinct. Consequently there exist polynomials of the form $p_j \circ \pi_j$ for $1 \leq j \leq n-1$ such that $(w_n \cdot \nabla)(p_j \circ \pi_j) = q_j \circ \pi_j$.

Thus $(w_n \cdot \nabla)(P - \sum_{j < n} p_j \circ \pi_j) \equiv 0$, so the polynomial $(P - \sum_{j < n} p_j \circ \pi_j)(x)$ is a function of $x_m = \pi_m(x)$ alone, and hence is of the form $p_n \circ \pi_n$. This proves that P is degenerate. \square

Lemma 3.2. *Let P be a homogeneous polynomial of some degree d . Then P is nondegenerate relative to a finite collection of subspaces $\{V_j\}$, of any dimension, if and only if there exists a constant-coefficient partial differential operator \mathcal{L} , homogeneous of degree d , such that $\mathcal{L}(P) \neq 0$ but $\mathcal{L}(p_j \circ \pi_j) \equiv 0$ for every polynomial $p_j : V_j \rightarrow \mathbb{C}$ of degree d .*

Proof. If such an operator \mathcal{L} exists then P is obviously nondegenerate. To prove the converse, fix d and denote by \mathcal{P}_d the vector space of all homogeneous polynomials of degree d . The pairing $\langle \mathcal{L}, P \rangle = \mathcal{L}(P)$ between homogeneous constant-coefficient differential operators of the same degree d and elements of \mathcal{P}_d is nondegenerate. Thus the dual space of \mathcal{P}_d may be canonically identified with the vector space of all such differential operators.

If P is nondegenerate then it does not belong to the subspace of all degenerate homogeneous polynomials, then there exists a linear functional which annihilates that subspace, but not P . This functional may be realized as $Q \mapsto \mathcal{L}(Q)$ for some operator \mathcal{L} , completing the proof in the converse direction. \square

Any polynomial may be expressed in a unique way as a sum of homogeneous polynomials of distinct degrees.

Lemma 3.3. *A polynomial P is nondegenerate relative to $\{V_j\}$ if and only if at least one of its homogeneous summands is nondegenerate. Moreover, a homogeneous polynomial is degenerate if and only if it may be expressed as $\sum_j p_j \circ \pi_j$ where each p_j is a homogeneous polynomial of the same degree.*

Proof. Both of these assertions follow from the simple fact that a polynomial $p_j : \mathbb{R}^\kappa \rightarrow \mathbb{R}$ is homogeneous of some degree if and only if $p_j \circ \pi_j : \mathbb{R}^k \rightarrow \mathbb{R}$ is homogeneous, of the same degree. \square

If P can be decomposed as $\sum_j f_j \circ \pi_j$ for some functions h_j which are not necessarily polynomials, then the decay bound (1.2) certainly fails, for the same reason noted earlier

(set $f_j = e^{-i\lambda h_j}$). The next lemma says that it makes no difference whether the h_j are taken to be polynomials, or arbitrary functions, in the definition of nondegeneracy.

Lemma 3.4. *A polynomial P is degenerate with respect to a collection of subspaces $\{V_j\}$ if and only if there exist distributions h_j in \mathbb{R}^κ such that $P = \sum_j h_j \circ \pi_j$ in some open set.*

Proof. Let $\{V_j\}$ be given, and let P be any nondegenerate polynomial of some degree d . We need only show that P can't be decomposed locally as $\sum_j h_j \circ \pi_j$, since the converse is a tautology. It is no loss of generality to suppose that the homogeneous summand of P of degree d is nondegenerate. For otherwise we may express that summand as $\sum_j p_j \circ \pi_j$ where the p_j are homogeneous polynomials of degree d , then replace P by $P - \sum_j p_j \circ \pi_j$ to reduce the degree.

By Lemma 3.2, there exists a constant-coefficient linear partial differential operator \mathcal{L} , homogeneous of degree d , such that $\mathcal{L}(P) \neq 0$, yet $\mathcal{L}(p_j \circ \pi_j) = 0$ for any polynomial p_j homogeneous of degree d , for any index j .

$\mathcal{L}(Q)(0) = 0$ for any polynomial $Q : \mathbb{R}^m \rightarrow \mathbb{R}$ which is homogeneous of any degree except d . By combining this with the hypothesis we find that $\mathcal{L}(p \circ \pi_j)(0) = 0$ for any polynomial $p : \mathbb{R}^\kappa \rightarrow \mathbb{R}$ and any index j . Consequently $\mathcal{L}(p \circ \pi_j)(x) = 0$ for all $x \in \mathbb{R}^m$. Since any distribution is a limit of polynomials in the distribution topology, $\mathcal{L}(h_j \circ \pi_j)$ likewise vanishes identically, in the sense of distributions. Yet \mathcal{L} does not annihilate P , so P cannot be decomposed as $\sum_j h_j \circ \pi_j$. \square

There is a characterization of nondegeneracy in terms of difference operators, analogous to that involving differential operators in Lemma 3.2, which will be used in the proof of Theorem 2.8. Let e_j denote the j -th coordinate vector in \mathbb{R}^m , define $\delta_j f(x) = f(x + e_j) - f(x)$, and define $\Delta_\alpha = \delta_1^{\alpha_1} \circ \cdots \circ \delta_m^{\alpha_m}$ where $\alpha = (\alpha_1, \dots, \alpha_m)$ with each α_j an arbitrary nonnegative integer. Define $|\alpha| = \sum_j \alpha_j$.

If $P : \mathbb{R}^m \rightarrow \mathbb{R}$ is a homogeneous polynomial of degree D and $|\alpha| = D$, then $\Delta_\alpha(P)$ is a constant. Thus there is a natural pairing between such the linear span of such difference operators, and such polynomials. This pairing is clearly nondegenerate, since whenever $|\alpha| = |\beta|$, $\Delta_\alpha(x^\beta) = 0$ if and only if $\alpha = \beta$. Thus it establishes an identification of the span of all Δ_α with $|\alpha| = D$ with the dual of the space of all homogeneous real polynomials of degree D .

Lemma 3.5. *Let $P : \mathbb{R}^m \rightarrow \mathbb{R}$ be a polynomial of degree D . Suppose that $P = P_D + R$ where P_D is homogeneous of degree D , R has degree $< D$, and P_D is nondegenerate with respect to a collection of subspaces $\{V_j\}$. Then there exist a finite subset $\{y_\beta\}$ of \mathbb{R}^m , and corresponding scalars $c_\beta \in \mathbb{R}$, such that for any $1 \leq j \leq n$ and any continuous function f_j , for all x and all $r > 0$,*

$$(3.1) \quad \sum_{\beta} c_\beta f_j \circ \pi_j(x + ry_\beta) = 0$$

but

$$(3.2) \quad \sum_{\beta} c_\beta P(x + ry_\beta) \equiv r^D.$$

Proof. By the preceding discussion there exist real scalars b_β such that $L = \sum_{|\beta|=D} b_\beta \Delta_\beta$ annihilates $p \circ \pi_j$ for any homogeneous polynomial p of degree D , for all j , but $L(P_D) \neq 0$. L of course annihilates $p \circ \pi_j$ whenever p has lower degree, and $L(p \circ \pi_j)(0) = 0$ whenever p is homogeneous of some higher degree. It follows that $L(\sum_j p_j \circ \pi_j)(x) = 0$ for all $x \in \mathbb{R}^m$,

for all polynomials p_j , and from this the same follows for all continuous functions p_j ; it even holds in the sense of distributions for arbitrary distributions p_j . Since R has degree $< D$, L also annihilates R , so $L(P)$ is a nonzero constant. By multiplying by a suitable scalar, we may arrange for this constant to be 1.

L takes the form $L(Q)(x) = \sum_{\beta} c_{\beta} Q(x + y_{\beta})$. Since P_D is homogeneous and any translate and dilate of R has degree $< D$, (3.2) follows. (3.1) for general r follows from the case $r = 1$ by scaling. \square

We say that a polynomial is a *monomial* if it can be expressed as a product of linear factors. By a *differential monomial* we mean a linear partial differential operator which can be expressed as a product of finitely many real vector fields, each with constant coefficients.

Lemma 3.6. *Let $\{V_j : 1 \leq j \leq n\}$ be a finite collection of subspaces of \mathbb{R}^m , each having dimension κ and lying in general position. If $(m - \kappa)n \leq m$ then any polynomial which is nondegenerate relative to $\{V_j\}$ is simply nondegenerate relative to $\{V_j\}$.*

Proof. Consider the variety $\mathcal{V} = \cup_j \mathcal{V}_j$ where $\mathcal{V}_j \subset \mathbb{R}^m$ is the $m - \kappa$ -dimensional subspace of \mathbb{R}^m equal to the range of π_j^* , where π_j^* is the adjoint of the orthogonal projection $\pi : \mathbb{R}^m \rightarrow V_j$. Let \mathcal{I} be the ideal in $\mathbb{C}[x]$ consisting of all real polynomials which vanish on \mathcal{V} . Then \mathcal{I} is of course finite-dimensional as a $\mathbb{C}[x]$ -module. According to the next lemma, the ideal \mathcal{I} is generated by some finite set of monomials.

Now consider the ring \mathcal{L} of all constant-coefficient linear partial differential operators in \mathbb{R}^m , and in it the ideal $\mathcal{L}_{\{V_j\}}$ consisting of all elements which annihilate all polynomials $p_j \circ \pi_j$, for all $1 \leq j \leq n$. The map $L \mapsto \sigma(L)$ from L to its symbol is an isomorphism of \mathcal{L} with $\mathbb{C}[x]$, which maps $\mathcal{L}_{\{V_j\}}$ to \mathcal{I} . Thus any element $L \in \mathcal{L}_{\{V_j\}}$ may be expressed as a finite sum $\sum_{\alpha} \ell_{\alpha} \circ \mathcal{M}_{\alpha}$ where each \mathcal{M}_{α} is a differential monomial which likewise annihilates all $p_j \circ \pi_j$, and the coefficients ℓ_{α} belong to \mathcal{L} .

If a polynomial P is nondegenerate relative to $\{V_j\}$, then there exists $L \in \mathcal{L}_{\{V_j\}}$ such that $L(P)$ does not vanish identically. Therefore there exists a differential monomial \mathcal{M}_{α} such that $\mathcal{M}_{\alpha}(P)$ does not vanish identically, so that P is simply nondegenerate. \square

Lemma 3.7. *Let $\mathcal{V} = \cup_{j=1}^n \mathcal{V}_j$ where each \mathcal{V}_j is a linear subspace of \mathbb{C}^m of dimension κ . Suppose that $(m - \kappa)n \leq m$, and that the subspaces \mathcal{V}_j lie in general position. Then the ideal $\mathcal{I} \subset \mathbb{C}[x]$ of all polynomials vanishing identically on \mathcal{V} is generated by monomials.*

Proof. Fix linear functions $y_{j,k}(x)$, for $1 \leq j \leq n$ and $1 \leq k \leq m - \kappa$, such that $\mathcal{V}_j = \{x : y_{j,k}(x) = 0 \text{ for all } 1 \leq k \leq m - \kappa\}$. The general position hypothesis means simply that these are linearly independent over \mathbb{C} . Thus after possibly adding some additional coordinates if $(m - \kappa)n < m$, we may regard these functions as coordinates on \mathbb{C}^m . Any monomial of the form $\prod_{j=1}^n y_{j,k(j)}$ belongs to \mathcal{I} .

Let $P \in \mathcal{I}$. We wish to show that P belongs to the ideal generated by all products $\prod_{j=1}^n y_{j,k(j)}$, where the function $j \mapsto k(j)$ runs over all $n^{m-\kappa}$ possibilities.

Since $P \equiv 0$ on \mathcal{V}_1 , it may be expressed as $P = \sum_{k=1}^{m-\kappa} y_{1,k} r_{1,k}$ for certain polynomials $r_{1,k}$, which have the additional property that $r_{1,k}$ is independent of $y_{1,i}$ for all $i < k$. Consider the restriction of P to the subspace where $y_{1,k} = 0$ for all $k < m - \kappa$. Since P vanishes on the intersection of this subspace with \mathcal{V}_2 , so must $y_{1,m-\kappa} r_{1,m-\kappa}$, and hence, by the general position hypothesis, so must $r_{1,m-\kappa}$. Therefore $r_{1,m-\kappa}$ must belong to the ideal generated by $\{y_{2,k}\}$.

Next consider the restriction of P to the subspace where $y_{1,k} = 0$ for all $k < m - \kappa - 1$. $r_{1,m-\kappa} \equiv 0$ on \mathcal{V}_2 , so by repeating the reasoning of the preceding paragraph we may conclude

that $r_{1,m-\kappa-1}$ belongs to the ideal generated by $\{y_{2,k}\}$. By descending induction on k we eventually find that every $r_{1,k}$ does also.

Thus we may write $P = \sum_{k_1=1}^{m-\kappa} \sum_{k_2=1}^{m-\kappa} y_{1,k_1} y_{2,k_2} r_{1,k_1,2,k_2}$ for certain polynomials $r_{1,k_1,2,k_2}$. By rearranging these we may also ensure that $r_{1,k_1,2,k_2}$ is independent of $y_{1,j}$ for all $j < k_1$, and of $y_{2,j}$ for all $j < k_2$.

The same reasoning may now be applied to \mathcal{V}_3 , and by induction to \mathcal{V}_j for all $j \leq n$. \square

Example. In \mathbb{R}^4 let $\mathcal{V}_1 = \{x : x_1 = x_2 = 0\}$, $\mathcal{V}_2 = \{x : x_3 = x_4 = 0\}$, $\mathcal{V}_3 = \{x : x_1 - x_3 = x_2 - x_4 = 0\}$. Thus $m = 4$, $\kappa = 2$, and $n = 3 > m/(m - \kappa)$. These three subspaces lie in general position. The polynomial $p(x) = x_1 x_4 - x_2 x_3$ belongs to the associated ideal \mathcal{I} . (p vanishes on other two-dimensional subspaces as well, but these together with the three subspaces listed do not lie in general position.) On the other hand, any monomial \mathcal{M} which belongs to \mathcal{I} must have degree at least three. Indeed, if \mathcal{M} has degree two then in order to vanish on $\mathcal{V}_1 \cup \mathcal{V}_2$, it must take the form $\mathcal{M}(x) = (ax_1 + bx_2)(cx_3 + dx_4)$. Then $\mathcal{M}(x_1, x_2, x_1, x_2)$ plainly cannot vanish identically on \mathcal{V}_3 .

4. THE SIMPLY NONDEGENERATE CASE

In proving (1.2), each function f_j may be assumed to be supported in a fixed compact set, the image under π_j of the support of the cutoff function η . We will assume this throughout the proofs of Theorems 2.1 and 2.3.

Proof of Theorem 2.3. The proof proceeds by induction on the degree n of multilinearity of Λ_λ , and is an adaptation of an argument in [1]. Adopt coordinates in which $V_n = \{(y, z) \in \mathbb{R}^{m-1} \times \mathbb{R}^1 : z = 0\}$ and hence $\pi_n(y, z) = y$. Write

$$\Lambda_\lambda(f_1, \dots, f_n) = \int f_n(y) \left(\int e^{i\lambda P(y,z)} \prod_{j=1}^{n-1} f_j(\pi_j(y, z)) \eta(y, z) dz \right) dy.$$

This equals $\langle T_\lambda(f_1, \dots, f_{n-1}), \overline{f_n} \rangle$ for a certain linear operator T_λ , whence

$$|\Lambda_\lambda(f_1, \dots, f_n)| \leq \|f_n\|_2 \|T_\lambda(f_1, \dots, f_{n-1})\|_2 \leq C \|f_n\|_\infty \|T_\lambda(f_1, \dots, f_{n-1})\|_2.$$

Thus it suffices to bound T_λ as an operator from $L^\infty \times \dots \times L^\infty$ to L^2 .

Now $\int |T_\lambda(f_1, \dots, f_{n-1})(y)|^2 dy$ equals

$$\int_{\mathbb{R}^{m-1}} \iint_{\mathbb{R}^2} e^{i\lambda(P(y,z) - P(y,z'))} \prod_{j < n} f_j(\pi_j(y, z)) \overline{f_j}(\pi_j(y, z')) \eta(y, z) \overline{\eta}(y, z') dz dz' dy.$$

Define $Q_\zeta(y, z) = P(y, z) - P(y, z + \zeta)$, and $\tilde{\eta}_\zeta(y, z) = \eta(y, z) \overline{\eta}(y, z + \zeta)$. Likewise define $F_j^\zeta : V_j \rightarrow \mathbb{C}$ so that $F_j^\zeta \circ \pi_j(y, z) = f_j(\pi_j(y, z)) \overline{f_j}(\pi_j(y, z + \zeta))$; the right-hand side is a function of $(\pi_j(y, z), \zeta)$ alone because of the linearity of π_j . Of course $\|F_j^\zeta\|_\infty \leq \|f_j\|_\infty^2$. With these definitions and the substitutions $x = (y, z)$, $z' = z + \zeta$,

$$\int |T_\lambda(f_1, \dots, f_{n-1})(y)|^2 dy = \int \Lambda_\lambda^\zeta(F_1^\zeta, \dots, F_{n-1}^\zeta) d\zeta$$

where

$$\Lambda_\lambda^\zeta(F_1^\zeta, \dots, F_{n-1}^\zeta) = \int_{\mathbb{R}^m} e^{i\lambda Q_\zeta(x)} \prod_{j=1}^{n-1} F_j^\zeta(\pi_j(x)) \tilde{\eta}_\zeta(x) dx.$$

The outer integral may be taken over a bounded subset of $\mathbb{R}^{m-\kappa}$, since $\tilde{\eta}_\zeta \equiv 0$ when $|\zeta|$ is sufficiently large. Henceforth we assume ζ to be restricted to such a bounded set.

For each ζ , consider the polynomial phase Q_ζ . We may assume that $|\lambda| \geq 1$, since (1.2) holds trivially otherwise. It is given as a hypothesis that there exists a differential operator of the form $L = \prod_{j=1}^n (w_j \cdot \nabla)$, with each $w_j \in V_j^\perp$, such that $\sup_{|x| \leq 1} |L(P)(x)| \geq c > 0$. Let $L' = \prod_{j < n} (w_j \cdot \nabla)$. For any $\rho \in (0, 1)$ define

$$E_\rho = \{\zeta : \max_{|x| \leq 1} |L'Q_\zeta(x)| \leq \rho\}.$$

For any $\zeta \notin E_\rho$, write $\lambda Q_\zeta = (\lambda\rho)\tilde{Q}$ where $\tilde{Q} = \rho^{-1}Q_\zeta$. Then by the induction hypothesis, there exist $C, \varepsilon' \in \mathbb{R}^+$ such that for all $\zeta \notin E_\rho$,

$$|\Lambda_\lambda^\zeta(F_1^\zeta, \dots, F_{n-1}^\zeta)| \leq C(1 + |\lambda|\rho)^{-\varepsilon'} \prod_{j < n} \|F_j^\zeta\|_\infty \leq C(1 + |\lambda|\rho)^{-\varepsilon'} \prod_{j < n} \|f_j\|_\infty^2.$$

The same bound holds for the integral over all $\zeta \notin E_\rho$, since ζ is confined to a bounded set.

For $\zeta \in E_\rho$ there is the trivial estimate

$$|\Lambda_\lambda^\zeta(F_1^\zeta, \dots, F_{n-1}^\zeta)| \leq C \prod_{j < n} \|F_j^\zeta\|_\infty,$$

so

$$\int_{E_\rho} |\Lambda_\lambda^\zeta(F_1^\zeta, \dots, F_{n-1}^\zeta)| d\zeta \leq C|E_\rho| \prod_{j < n} \|f_j\|_\infty^2.$$

Now

$$(4.1) \quad |E_\rho| \leq C\rho^\delta \text{ for some } \delta > 0 \text{ and } C < \infty.$$

Indeed, by hypothesis $\sup_{(x, \zeta)} |\partial_\zeta(L'Q_\zeta(x))| \geq c > 0$, and as is well known, this implies that if (x, ζ) is restricted to any fixed bounded set, then

$$(4.2) \quad |\{(x, \zeta) : |L'Q_\zeta(x)| \leq r\}| \leq Cr^a$$

for some $a > 0$ where C, a depend only on c and on an upper bound for the degree of Q as a polynomial in (x, ζ) . In particular, so long as x, ζ are restricted to lie in any fixed bounded set,

$$|\{\zeta : \max_x |L'Q_\zeta(x)| \leq r\}| \leq Cr^a.$$

Thus in all

$$\int |\Lambda_\lambda^\zeta(F_1^\zeta, \dots, F_{n-1}^\zeta)| d\zeta \leq C[(|\lambda|\rho)^{-\varepsilon'} + \rho^\delta] \prod_{j < n} \|f_j\|_\infty^2.$$

Choosing $\rho = |\lambda|^{-c}$ for any fixed $c \in (0, 1)$ yields the desired bound. \square

5. THE POWER DECAY PROPERTY FOR $\kappa = 1$

This section is devoted to the proof of Theorem 2.1. The proof turns on a concept related to a notion of uniformity employed by Gowers [2]. Let $d \geq 1$, and fix a bounded ball $B \subset \mathbb{R}^m$. Let $\tau > 0$ be a small quantity to be chosen below, let $|\lambda| \geq 1$, and consider any function $f \in L^2(\mathbb{R}^m)$ supported in B .

Definition 5.1. f is λ -nonuniform if there exist a polynomial q of degree $\leq d$ and a scalar c such that

$$(5.1) \quad \|f - ce^{iq}\|_{L^2(B)} \leq (1 - |\lambda|^{-\tau})\|f\|_{L^2}.$$

Otherwise f is said to be λ -uniform.

This notion depends on the parameters d, τ . So long as they remain fixed there exists $C < \infty$, depending also on B , such that any λ -uniform function $f \in L^2(B)$ satisfies favorable bounds for generalized Fourier coefficients:

$$(5.2) \quad \left| \int f(t) e^{-iq(t)} dt \right| \leq C |\lambda|^{-\tau/2} \|f\|_{L^2(B)}$$

uniformly for all real-valued polynomials q of degree $\leq d$. Indeed, f could otherwise be decomposed in $L^2(B)$ into its projection onto e^{iq} plus an orthogonal vector, implying (5.1).

The proof of Theorem 2.1 will proceed by induction on n , the number of subspaces $\{V_j\}$. The inductive step enters in the following way: If $f_1 = e^{ip}$ for some polynomial p , then $\Lambda_\lambda(e^{ip}, f_2, \dots, f_n) = \int e^{i\tilde{P}(x)} \prod_{j=2}^n f_j(\pi_j(x)) \eta(x) dx$ where $\tilde{P} = P + p \circ \pi_1$. With p held fixed, this may be regarded as an $(n-1)$ -multilinear operator acting on (f_2, \dots, f_n) , with the new phase \tilde{P} and the smaller collection of subspaces $\{V_j : 2 \leq j \leq n\}$. \tilde{P} is nondegenerate relative to this subcollection; moreover, this nondegeneracy is uniform as p varies over all polynomials of uniformly bounded degrees. Thus it is a consequence of the inductive hypothesis that

$$(5.3) \quad |\Lambda_\lambda(e^{ip}, f_2, \dots, f_n)| \leq C |\lambda|^{-\varepsilon}$$

provided that $\|f_j\|_2 \leq 1$ for all $2 \leq j \leq n$, uniformly for all polynomials p of uniformly bounded degrees.

Proof of Theorem 2.1. Given the cutoff function η , there exist intervals B_j of finite lengths in \mathbb{R}^1 such that $\Lambda_\lambda(f_1, \dots, f_n)$ depends only on the restriction of each f_j to B_j , so henceforth we assume f_j to be supported in B_j . Thus $\|f_j\|_{L^2}$ equals $\|f_j\|_{L^2(B_j)}$.

We may assume henceforth² that n is strictly larger than m , since the theorem is already known in a more precise form [8],[9],[10] for the case $n = m$. Let e_1 be a unit vector orthogonal to the span of $\{V_j : 2 \leq j \leq m\}$. Since the subspaces $\{V_j : 2 \leq j \leq m\}$ span a space of codimension one, e_1 is uniquely determined modulo multiplication by -1 , and it cannot be orthogonal to V_1 because of the general position hypothesis.

Likewise choose some unit vector e_2 orthogonal to $\text{span}(\{V_j : j > m\})$, and not orthogonal to V_1 . $\cup_{j>m} V_j$ spans a space of dimension $n - m < m$ by the assumption that $n < 2m$ and the general position hypothesis, and the only way that all unit vectors in its orthocomplement could be forced to be orthogonal to V_1 is if V_1 were to be contained in $\text{span}(\{V_j : j > m\})$. But this cannot happen, again by general position and the restriction $n - m < m$. Thus there exists at least one vector e_2 with the required properties.

e_2 cannot be orthogonal to $\text{span}(\{V_j : 2 \leq j \leq m\})$, since $\cup_{j=2}^m V_j$ spans \mathbb{R}^m by the general position assumption ($n > m$). e_1 cannot be orthogonal to $\text{span}(\{V_j : j > m\})$, since then it would be orthogonal to $\cup_{j \geq 2} V_j$, which we have just seen to be impossible. Thus e_2 is automatically linearly independent of e_1 .

Given $\{V_j\}$, P , a cutoff function η , and λ , define $A(\lambda)$ to be the best constant (that is, the infimum of all admissible constants) in the inequality

$$(5.4) \quad |\Lambda_\lambda(f_1, \dots, f_n)| \leq A(\lambda) \prod_j \|f_j\|_{L^2}.$$

As noted before the statement of Theorem 2.1, the hypotheses of general position and $n \leq 2m$ ensure that A_λ is finite for all λ . We will assume that $\|f_j\|_{L^2} \leq 1$ for all j , and

²Our proof can be adapted to the simpler case $n \leq m$ as well.

that $|\lambda|$ exceeds some sufficiently large constant. To prove the theorem it suffices to obtain a bound of $C|\lambda|^{-\varepsilon}$ under these additional hypotheses.

The analysis of $\Lambda_\lambda(f_1, \dots, f_n)$ is divided into two cases, depending on whether or not f_1 is λ -uniform. If it is not, let $f = f_1, c, q$ satisfy (5.1). Then

$$|\Lambda_\lambda(f_1 - ce^{iq}, f_2, \dots, f_n)| \leq A(\lambda)(1 - |\lambda|^{-\tau}).$$

Moreover $|c|$ is majorized by an absolute constant since $\|f_1\|_{L^2} = 1$ and e^{iq} is unimodular, so by the inductive hypothesis (5.3)

$$|\Lambda_\lambda(ce^{iq}, f_2, \dots, f_n)| \leq C|\Lambda_\lambda(e^{iq}, f_2, \dots, f_n)| \leq C|\lambda|^{-\sigma}$$

for certain $C, \sigma \in (0, \infty)$; this bound holds uniformly provided that P varies over a set of uniformly nondegenerate polynomials. Combining the two terms yields

$$(5.5) \quad |\Lambda_\lambda(f_1, \dots, f_n)| \leq A(\lambda)(1 - |\lambda|^{-\tau}) + |\lambda|^{-\sigma}$$

provided that $\|f_j\|_{L^2} \leq 1$ for all j , and that f_1 is λ -nonuniform.

Suppose next that f_1 is λ -uniform. Adopt coordinates $(t, y) \in \mathbb{R}^2 \times \mathbb{R}^{m-2}$ where \mathbb{R}^2 is the span of e_1, e_2 and \mathbb{R}^{m-2} is its orthocomplement, by writing $x = t_1 e_1 + t_2 e_2 + y$. Let $P^y(t) = P(t, y)$. Define

$$F_1^y(t_2) = \prod_{j=2}^m f_j \circ \pi_j(t, y);$$

the right-hand side is independent of t_1 since e_1 is orthogonal to V_j for all $2 \leq j \leq m$. Likewise define

$$F_2^y(t_1) = \prod_{j=m+1}^n f_j \circ \pi_j(t, y),$$

which is independent of t_2 . In the same way, since neither e_1 nor e_2 is orthogonal to V_1 , $f_1 \circ \pi_1(t, y)$ depends for each y only on a certain projection $\pi(t)$ of $t \in \mathbb{R}^2$ onto \mathbb{R}^1 , so we may write

$$G^y(\pi(t)) = f_1 \circ \pi_1(t, y).$$

The general position hypothesis, the hypothesis $n \leq 2m$, and Fubini's theorem together imply that

$$\begin{aligned} \int \|F_1^y\|_{L^2}^2 \|G^y\|_{L^2}^2 dy &= C \prod_{j=1}^m \|f_j\|_{L^2}^2 \\ \int \|F_2^y\|_{L^2}^2 dy &= C \|f_1\|_{L^2}^2 \prod_{j=m+1}^n \|f_j\|_{L^2}^2, \end{aligned}$$

and consequently

$$(5.6) \quad \int \|F_1^y\|_{L^2} \|F_2^y\|_{L^2} \|G^y\|_{L^2} dy \leq C \prod_{j=1}^n \|f_j\|_{L^2}$$

by Cauchy-Schwarz.

For each $y \in \mathbb{R}^{m-2}$ consider

$$\Lambda_\lambda^y = \int_{\mathbb{R}^2} e^{i\lambda P^y(t)} F_1^y(t_2) F_2^y(t_1) G^y(\pi(t)) \eta(t, y) dt.$$

Define the set $\mathcal{B} \subset \mathbb{R}^{m-2}$ of bad parameters to be the set of all y for which P^y may be decomposed as

$$P^y(t) = Q_1(t_1) + Q_2(t_2) + Q_3(\pi(t)) + R(t)$$

where Q_j are real-valued polynomials of degrees $\leq d$ on \mathbb{R}^1 for $j = 1, 2$, t_j is a certain linear function of t , $R : \mathbb{R}^2 \rightarrow \mathbb{R}$ is likewise a real-valued polynomial of degree $\leq d$, and

$$\|R\| \leq |\lambda|^{-\rho};$$

that is, P^y has small norm in the quotient space of polynomials modulo degenerate polynomials, relative to the three projections $t \mapsto t_1, t_2, \pi(t)$. Here $\|\cdot\|$ denotes any fixed norm on the vector space of all polynomials of degree $\leq d$, and $\rho \in (0, 1)$ is another parameter to be specified. If $y \notin \mathcal{B}$ then $|\lambda|^\rho P^y$ is at least a fixed positive distance from the span of all polynomials $Q_1(t_1) + Q_2(t_2) + Q_3(\pi(t))$, so we may apply Theorem 2.3 with $n = 3$, $m = 2$, and the phase $|\lambda|^\rho P^y$, to obtain

$$(5.7) \quad |\Lambda_\lambda^y| \leq C(|\lambda|^{1-\rho})^{-\tilde{\rho}} \|F_1^y\|_{L^2} \|F_2^y\|_{L^2} \|G^y\|_{L^2} \text{ for some } C, \tilde{\rho} \in \mathbb{R}^+.$$

This together with (5.6) implies that

$$\int_{y \notin \mathcal{B}} |\Lambda_\lambda^y| dy \leq C |\lambda|^{-(1-\rho)\tilde{\rho}},$$

as desired.

Bad parameters cannot be handled in this way. Nor is it true that the set of bad parameters has measure $O(|\lambda|^{-\delta})$ for some $\delta > 0$; it could happen that every parameter is bad. Nonetheless we claim that if the exponent ρ appearing in the definition of bad parameters is chosen to be sufficiently small, then there exists $\varepsilon > 0$ such that uniformly for all $y \in \mathcal{B}$,

$$(5.8) \quad |\Lambda_\lambda^y| \leq C |\lambda|^{-\varepsilon} \|F_1^y\|_{L^2} \|F_2^y\|_{L^2},$$

which by (5.6) again implies the desired bound. To verify this fix $y \in \mathcal{B}$, and let $P^y(t) = Q_1(t_1) + Q_2(t_2) + Q_3(\pi(t)) + R(t)$ as above. Write $\pi(t) = c_1 t_1 + c_2 t_2$ where c_1, c_2 are both nonzero. Introduce the functions

$$\begin{aligned} \tilde{F}_1(t_2) &= F_1^y(t_2) e^{i\lambda Q_2(t_2)}, \\ \tilde{F}_2(t_1) &= F_2^y(t_1) e^{i\lambda Q_1(t_1)}, \\ \tilde{G}(s) &= G^y(s) e^{i\lambda Q_3(s)}, \\ \zeta(t) &= \eta(t, y) e^{i\lambda R(t)}. \end{aligned}$$

Since λR is a polynomial of bounded degree which is $O(|\lambda|^{1-\rho})$ on the support of ζ , the same holds for all its derivatives and therefore

$$|\widehat{\zeta}(\xi)| \leq C_N |\lambda|^{1-\rho} (1 + |\xi|)^{-N} \text{ for all } \xi \text{ and all } N.$$

By Fourier inversion,

$$\Lambda_\lambda^y = c \int_{\mathbb{R}^3} \widehat{\tilde{G}}(\xi_0) \widehat{\tilde{F}_1}(-c_2 \xi_0 - \xi_2) \widehat{\tilde{F}_2}(-c_1 \xi_0 - \xi_1) \widehat{\zeta}(\xi_1, \xi_2) d\xi_0 d\xi_1 d\xi_2.$$

From this representation, the bound for $\widehat{\zeta}$, and Plancherel's theorem it follows that

$$|\Lambda_\lambda^y| \leq C |\lambda|^{1-\rho} \|\widehat{\tilde{G}}\|_{L^\infty} \|\tilde{F}_1\|_{L^2} \|\tilde{F}_2\|_{L^2}.$$

Here $\|\tilde{F}_i\|_{L^2} = \|F_i^y\|_{L^2}$. \tilde{G} is obtained from f_1 by composing with a fixed linear mapping, translating by an arbitrary amount, and multiplying with e^{iq} for some real-valued polynomial q of degree $\leq d$. Therefore (5.2) implies that

$$\|\widehat{\tilde{G}}\|_{L^\infty} \leq C |\lambda|^{-\tau}$$

for some $C, \tau \in \mathbb{R}^+$ independent of y . Thus if f_1 is λ -uniform then

$$|\Lambda_\lambda^y| \leq C|\lambda|^{1-\rho-\tau} \|F_1^y\|_{L^2} \|F_2^y\|_{L^2} \leq C|\lambda|^{1-\rho-\tau}.$$

The parameter $\rho \in (0, 1)$ is at our disposal; we choose $\rho < 1$ sufficiently close to 1 such that $1 - \rho - \tau = -\varepsilon$ is strictly negative. This completes the analysis of the case where f_1 is λ -uniform.

Combined with (5.5) and (5.7), this proves that

$$(5.9) \quad A(\lambda) \leq C \max(|\lambda|^{-\varepsilon}, |\lambda|^{-(1-\rho)\tilde{\rho}}, A(\lambda)(1 - |\lambda|^{-\tau}) + |\lambda|^{-\sigma}).$$

Therefore $A(\lambda)$ is majorized by a constant times some negative power of $|\lambda|$, as was to be proved. \square

6. ANALYSIS OF OSCILLATORY SINGULAR INTEGRAL OPERATORS

In this section we combine the results on nonsingular oscillatory integral operators proved above with the currently existing theory of multilinear Calderón-Zygmund singular integral operators to prove Theorems 2.6 and 2.7. Let $d, k, n \geq 1$, and let K be a Calderón-Zygmund distribution in \mathbb{R}^k . Thus K is a tempered distribution which agrees with a Lipschitz function on $\mathbb{R}^k \setminus \{0\}$, and $K(ry) \equiv r^{-k}y$ in the sense of distributions. For $1 \leq j \leq n$ let $\ell_j : \mathbb{R}^k \rightarrow \mathbb{R}^d$ be a linear mapping, and suppose that the intersection of the nullspaces of all the ℓ_j is $\{0\}$. Consider the multilinear operators

$$(6.1) \quad T(f_1, \dots, f_n)(x) = \langle K(y), \prod_{j=1}^n f_j(x + \ell_j(y)) \rangle$$

where $f_j \in C_0^\infty$, and the pairing is that of the distribution K with the test function $y \mapsto \prod_{j=1}^n f_j(x + \ell_j(y))$. To any real-valued polynomial $P(x, y)$ defined on \mathbb{R}^{d+k} is associated the operator

$$(6.2) \quad T_P(f_1, \dots, f_n)(x) = \langle e^{iP(x, y)} K(y), \prod_{j=1}^n f_j(x + \ell_j(y)) \rangle.$$

It is not known under what conditions operators T act boundedly on Lebesgue spaces, but we can assert a conditional result, generalizing Theorem 2.7. Set $\ell_0(y) \equiv 0$, and in \mathbb{R}^{d+k} let V_j be the orthocomplement of the nullspace of the mapping $(x, y) \mapsto x + \ell_j(y)$. Thus $x + \ell_j(y)$ may alternatively be written as $\pi_j(x, y)$.

Theorem 6.1. *Let $d, n, n, K, T, \{\ell_j\}$ be as above. Suppose that T maps $\otimes_{j=1}^n L^{p_j}$ boundedly to L^q for some exponents $1 < p_j < \infty$ and $0 < q < \infty$ which satisfy $q^{-1} = \sum_j p_j^{-1}$. Suppose furthermore that the associated subspaces $\{V_j : 0 \leq j \leq n\}$ have the uniform decay property. Then for any real-valued polynomial P , T_P maps $\otimes_{j=1}^n L^{p_j}$ boundedly to L^q . Moreover for any D there exists $C_D < \infty$ such that $\|T_P(f_1, \dots, f_n)\|_q \leq CC_D \prod_j \|f_j\|_{p_j}$ uniformly for all real-valued polynomials P of degrees $\leq D$, where C is the operator norm of T from $\otimes_j L^{p_j}$ to L^q .*

If $\{V_j\}$ merely has the uniform decay property for all polynomials of degrees $\leq D$, then the conclusion of the theorem holds for polynomials of degrees $\leq D$.

A sufficient condition for T to be bounded was established in [6]. Combining it with Theorem 6.1 gives Theorem 2.7, and its special case Theorem 2.6.

The proof of Theorem 6.1 is little different from its linear prototype [9]. One step in the proof requires passing to associated truncated operators. This can be done quite

generally by means of the following lemma, versions of which have appeared elsewhere in the literature.

Lemma 6.2. *Let $m \geq \kappa \geq 1$ and $n \geq 0$ be integers. For $0 \leq j \leq n$ let $\ell_j : \mathbb{R}^m \rightarrow \mathbb{R}^\kappa$ be linear mappings, the intersection of all of whose nullspaces is $\{0\}$. Let $p_j \in [1, \infty]$.*

Let $\eta \in C_0^{m+1}(\mathbb{R}^m)$ have compact support. There exists $A < \infty$, depending only on the C^{m+1} norm and support of η , with the following property. Let \mathcal{K} be any tempered distribution in \mathbb{R}^m , and suppose there exists $C < \infty$ such that

$$(6.3) \quad \left| \langle \mathcal{K}, \prod_{j=0}^n f_j \circ \ell_j \rangle \right| \leq C \prod_j \|f_j\|_{p_j} \quad \text{for all } f_j \in C_0^\infty(\mathbb{R}^\kappa).$$

Then

$$(6.4) \quad |\langle \eta \mathcal{K}, \prod_{j=0}^n f_j \circ \ell_j \rangle| \leq AC \prod_j \|f_j\|_{p_j} \quad \text{for all } f_j \in C_0^\infty(\mathbb{R}^\kappa).$$

Here $\langle \mathcal{K}, \cdot \rangle$ denotes the pairing of distribution with test function; the hypothesis that the intersection of the nullspaces is trivial guarantees that $\prod_{j=0}^n f_j \circ \ell_j$ is compactly supported.

Remark. The corresponding conclusion holds for multilinear *operators* mapping $\otimes_{j=1}^n L^{p_j}$ to L^q , for any $q \in (0, \infty]$, with the sole modification that when $q < 1$, the hypothesis $\eta \in C_0^{m+1}$ should be changed to $\eta \in C_0^s$ for some suitably large $s = s(q, m)$. This follows from the argument below, together with the quasi-triangle inequality $\|f+g\|_q^q \leq \|f\|_q^q + \|g\|_q^q$.

Proof. Write $\mathcal{T}_{\mathcal{K}}(f_0, \dots, f_n) = \langle \mathcal{K}, \prod_j f_j \circ \ell_j \rangle$. Fix $\zeta \in C_0^\infty(\mathbb{R}^\kappa)$ such that $\zeta \circ \ell_j(y) \equiv 1$ for all y in the support of η . Thus

$$\mathcal{T}_{\eta \mathcal{K}}(f_0, \dots, f_n) = \langle \mathcal{K} \eta, \prod_{j=0}^n (f_j \cdot \zeta) \circ \ell_j \rangle.$$

Expand

$$\eta(y) \prod_{j=0}^n \zeta \circ \ell_j(y) = \int_{\xi \in \mathbb{R}^m} a(\xi) e^{iy \cdot \xi} d\xi$$

where $a(\xi) = O((1 + |\xi|)^{-m-1})$. It suffices to prove that

$$\left| \langle \mathcal{K}, e^{iy \cdot \xi} \prod_j f_j \circ \ell_j(y) \rangle \right| \leq AC \prod_j \|f_j\|_{p_j}.$$

But since the intersection of the nullspaces of $\{\ell_j\}$ is trivial, there exist linear mappings $L_j : \mathbb{R}^\kappa \rightarrow \mathbb{R}^m$ such that $y \equiv \sum_{j=0}^n L_j \circ \ell_j(y)$ for all $y \in \mathbb{R}^m$. Consequently $e^{iy \cdot \xi}$ may be rewritten as $\prod_j e^{i \ell_j(y) \cdot L_j^*(\xi)}$, and if we define $\tilde{f}_j(x) = f_j(x) e^{ix \cdot L_j^*(\xi)}$ then our functional becomes $\langle \mathcal{K}, \prod_j \tilde{f}_j \circ \ell_j \rangle$. Since $\|\tilde{f}_j\|_{p_j} = \|f_j\|_{p_j}$, (6.4) follows from (6.3). \square

Proof of Theorem 6.1. Let subspaces $\{V_j\}$, a degree $D \geq 1$, and a polynomial P of degree $\leq D$ be given. We will prove the result only for $q \geq 1$, which permits the use of duality and thus simplifies notation, leaving the simple modifications for $q < 1$ to the reader. Let p_0 be the exponent dual to q , and recall that $\ell_0(y) \equiv 0$. By duality, matters reduce to a multilinear functional

$$\mathcal{T}_P(f_0, \dots, f_n) = \left\langle K(y) e^{iP(x, y)}, \prod_{j=0}^n f_j(x + \ell_j(y)) \right\rangle,$$

where the pairing is with respect to (x, y) . We proceed by induction on D , the result for $D = 0$ being given as a hypothesis.

Decompose $P = P_D + R$ where R has degree $< D$ and P_D is homogeneous of degree D . If P_D is degenerate then by decomposing $P_D = \sum_j q_j(x + \ell_j(y))$ for certain real-valued polynomials q_j , we may rewrite $\mathcal{T}_P(f_0, \dots, f_n) = \mathcal{T}_R(e^{iq_0} f_0, \dots, e^{iq_n} f_n)$. By induction on the degree, \mathcal{T}_R satisfies the desired estimate, which is equivalent to the desired estimate for \mathcal{T}_P since $\|e^{iq_j} f_j\|_{p_j} = \|f_j\|_{p_j}$.

If P_D is nondegenerate, there exists a smallest integer N such that $P_D(2^N x, 2^N y)$ has norm ≥ 1 in the quotient space of homogeneous polynomials of degree D modulo degenerate homogeneous polynomials of that degree, with respect to some fixed choice of norm on that finite-dimensional space. This norm is then also $\leq 2^D$.

By rescaling the variables x, y by a factor of 2^N we may reduce henceforth to the case $N = 0$. Because K is homogeneous of the critical degree and $\sum_{j=0}^n p_j^{-1} = 1$, such a rescaling does not affect the inequality to be proved.

By replacing P_D by $P_D - \sum_j q_j(x + \ell_j(y))$ for appropriate homogeneous real-valued polynomials q_j of degree D , we may assume henceforth that P_D has norm $\lesssim 1$ in the space of all homogeneous polynomials of degree D , rather than merely in the quotient space.

Fix a cutoff function $\zeta \in C_0^\infty(\mathbb{R}^k)$ which is $\equiv 1$ in some neighborhood of the origin, and consider the truncated multilinear functional $\mathcal{T}'_P = \langle \zeta(y) K e^{iP(x,y)}, \prod_{j=0}^n f_j(x + \ell_j(y)) \rangle$. Introduce also $\eta \in C_0^\infty(\mathbb{R}^d)$ such that $\sum_{\nu \in \mathbb{Z}^d} \eta(x - \nu) \equiv 1$ and decompose

$$\mathcal{T}'_P(f_0, \dots, f_n) = \sum_{\nu=(\nu_0, \dots, \nu_n) \in \mathbb{Z}^{(n+1)d}} \langle \zeta(y) K e^{iP(x,y)}, \prod_{j=0}^n f_j(x + \ell_j(y)) \eta(x + \ell_j(y) - \nu_j) \rangle.$$

Because of the presence of the compactly supported factor $\zeta(y)$, there exists $C_0 < \infty$ such that any term with parameter ν vanishes unless $\max_j |\nu_j - \nu_0| \leq C_0$. Therefore by the triangle inequality, Hölder's inequality and the hypothesis $\sum_j p_j^{-1} = 1$, it suffices to prove the desired bound for fixed ν , with f_j replaced by $f_j \eta(x + \ell_j(y) - \nu_j)$, so long as a majorization uniform in ν is obtained.

Make the change of variables $x = z + \nu_0$. $P(x, y) = P_D(x, y) + R(x, y) = P_D(z, y) + \tilde{R}_{\nu_0}(z, y)$ where \tilde{R}_{ν_0} has degree $< D$. Moreover, if another cutoff function $\tilde{\zeta} \in C_0^\infty$ is chosen to be $\equiv 1$ in a sufficiently large neighborhood of the origin, then our functional may be written as

$$\langle K(y) e^{i\tilde{R}_{\nu_0}(z,y)} (e^{iP_D(z,y)} \zeta(y) \tilde{\zeta}(z, y)), \prod_{j=0}^n \tilde{f}_j(z + \ell_j(y)) \rangle$$

where each \tilde{f}_j is an appropriate translate of the product of f_j with $\eta(x + \ell_j(y) - \nu_j)$. Now the factor $\zeta(y) e^{iP_D(z,y)} \tilde{\zeta}(z, y)$ is smooth and compactly supported as a function of (z, y) , and is bounded above in any C^s norm by a constant depending only on s, D and the choice of $\tilde{\zeta}$. By induction on D , the functional $\langle K e^{i\tilde{R}_{\nu_0}(z,y)}, \prod_{j=0}^n \tilde{f}_j(z + \ell_j(y)) \rangle$ maps $\otimes_j L^{p_j}$ to \mathbb{C} boundedly, uniformly for all polynomials \tilde{R}_{ν_0} of degree $< D$. Therefore we may invoke Lemma 6.2 to conclude that

$$\left| \langle K e^{i\tilde{R}_{\nu_0}(z,y)} (e^{iP_D(z,y)} \zeta(y) \tilde{\zeta}(z, y)), \prod_{j=0}^n \tilde{f}_j(z + \ell_j(y)) \rangle \right| \leq C \prod_j \|f_j\|_{p_j},$$

uniformly in ν .

It remains to analyze

$$\langle Ke^{iP}(1 - \zeta(y)), \prod_j f_j(x + \ell_j(y)) \rangle = \sum_{r=0}^{\infty} \langle Ke^{iP}h(2^{-r}y), \prod_j f_j(x + \ell_j(y)) \rangle$$

where $h(y) = \zeta(y/2) - \zeta(y)$. We claim that the r -th summand is majorized by $C2^{-\varepsilon r} \prod_j \|f_j\|_{p_j}$, for some $\varepsilon > 0$ and $C < \infty$ depending only on $D, \{V_j\}$. Because of the homogeneity of K and the condition $\sum_j p_j^{-1} = 1$, this is equivalent to

$$(6.5) \quad |\langle K(y)h(y)e^{iP_r}, \prod_j f_j(x + \ell_j(y)) \rangle| \leq C2^{-\varepsilon r} \prod_j \|f_j\|_{p_j},$$

via the substitution $(x, y) = (2^r x', 2^r y')$, where $P_r(x, y) = P(2^r x, 2^r y) = 2^{Dr} P_D(x, y) + \tilde{R}_r(x, y)$ where \tilde{R} has degree $< D$ and is real-valued.

As above, we may reduce matters to the case where all f_j are supported in a fixed bounded set, by introducing a partition of unity. \tilde{R}_r is thereby further modified and depends also on the index ν_0 ; it satisfies no useful upper or lower bounds but still has degree strictly $< D$.

Any such polynomial $2^{Dr} P_D + \tilde{R}$, where \tilde{R} has degree $< D$, has norm $\geq c2^{Dr}$ in the quotient space of polynomials of degree $\leq D$ modulo degenerate polynomials of degree $\leq D$, for some fixed constant $c > 0$, depending only on D . The function Kh is Lipschitz, since K is a Calderón-Zygmund kernel and h vanishes identically in some neighborhood of the origin. Therefore (6.5) follows from the assumption that $\{V_j\}$ has the uniform decay property. \square

7. SUBLEVEL SETS

We next prove Theorem 2.8. The argument requires an elaboration of the characterization of nondegeneracy in terms of difference operators that was established in Lemma 3.5.

Define divided difference operators

$$\begin{aligned} \delta_{j,r} f(x) &= [f(x + r e_j) - f(x)]/r, \\ \Delta_{\alpha,r} &= \delta_{1,r_{1,1}} \circ \delta_{1,r_{1,2}} \cdots \circ \delta_{1,r_{1,\alpha_1}} \circ \cdots \circ \delta_{n,r_{n,1}} \circ \cdots \circ \delta_{n,r_{n,\alpha_n}} \end{aligned}$$

where in the second definition r has arbitrary components $r_{j,k} \in \mathbb{R}^+$ for all $1 \leq j \leq m$ and $1 \leq k \leq \alpha_j$.

Let subspaces $V_j \subset \mathbb{R}^m$ of some dimension κ be given for $1 \leq j \leq n$, where n, m, κ are arbitrary. Let P be any polynomial which is nondegenerate relative to $\{V_j\}$. Let D be the degree of P . As already shown, we may suppose without loss of generality that the homogeneous component of P of degree D is itself nondegenerate.

Let $L = \sum_{|\beta|=D} b_\beta \Delta_\beta$ be the homogeneous difference operator constructed in Lemma 3.5, which annihilates all degenerate polynomials $p_j \circ \pi_j$, but does not annihilates P . Consider the more general operators

$$(7.1) \quad L_r = \sum_{|\beta|=D} b_\beta \prod_{j,k} \Delta_{\beta,r_{\beta}}$$

where there is associated to each index β a separate multi-parameter r_β whose components are $r_{\beta,j,k} \in \mathbb{R}^+$. The proof of Lemma 3.5 demonstrates that

$$(7.2) \quad L_r(P - \sum_j f_j \circ \pi_j) \equiv 1 \text{ a.e. for all } f_j \in L^1_{\text{loc}},$$

for all multi-parameters r . (We require that f_j be locally integrable, rather than merely measurable, in order that it can be interpreted as a distribution.) Indeed, for any homogeneous polynomial Q of degree D , $L_r Q(0) = LQ(0)$ for any r , by homogeneity because we are working with *divided* difference operators. On the other hand, $L_r Q(0) = 0$ for any homogeneous polynomial of any other degree. The rest of the proof parallels that of Lemma 3.5.

For any function g , point x , multi-index β and multi-parameter r_β , $\Delta_{\beta, r_\beta} g(x)$ is a linear combination of $2^{|\beta|}$ values $g(x + \sum_i r_{\beta, i} \sigma_{\beta, i} y_{\beta, i})$, where i ranges over an index set of cardinality $|\beta| = D$, each $y_{\beta, i}$ belongs to \mathbb{R}^m , and each $\sigma_{\beta, i} \in \{0, 1\}$, with one summand for each of the 2^D pairs $(i, \sigma_{\beta, i})$. Moreover, the coefficients in this linear combination are fixed constants, depending on $(\beta, i, \sigma_{\beta, i})$, times $\prod_i r_{\beta, i}^{-1}$. Each $y_{\beta, i}$ is in fact one of the m unit coordinate vectors in \mathbb{R}^m , but this information will not be used in the further discussion.

L_r involves a further summation over β , so that $L_r f(x)$ is a linear combination of values of f at the points of $x + \cup_\beta \{\sum_i r_{\beta, i} \sigma_{\beta, i} y_{\beta, i}\}$. It will be convenient for the reasoning below to simplify the form of this linear combination by placing the variables β, i on a more even footing. To achieve this we enlarge the collection of points $\cup_\beta \{\sum_i r_{\beta, i} \sigma_{\beta, i} y_{\beta, i}\}$, whose cardinality is $N + 2^D$ where N is the number of indices β , by forming the finite set

$$(7.3) \quad Y_r = \left\{ \sum_\beta \sum_i r_{\beta, i} \sigma_{\beta, i} y_{\beta, i} : \text{each } \sigma_{\beta, i} \in \{0, 1\} \right\},$$

whose cardinality is 2^{N+D} . With this notation, L_r takes the form

$$(7.4) \quad L_r g(x) = \sum_{y \in Y_r} c_{y, r} g(x + y)$$

where each coefficient $c_{y, r}$ is proportional to the product of reciprocals of certain components of r , while Y_r takes the form $Y_r = \{\sum_{\alpha \in \mathcal{A}} r_\alpha \sigma_\alpha y_\alpha : \sigma \in \{0, 1\}^{|\mathcal{A}|}\}$ for a certain index set \mathcal{A} and points y_α . For typical (non-monomial) L_r , the overwhelming majority of the coefficients $c_{y, r}$ will equal zero.

Given P and functions f_j , let

$$(7.5) \quad E_\varepsilon = \left\{ x \in \mathbb{R}^k : |P(x) - \sum_j f_j(\pi_j(x))| < \varepsilon \right\}.$$

In the spirit of [1], we now describe a certain combinatorial restriction on these sublevel sets which is implied by nondegeneracy of P .

Lemma 7.1. *Suppose that a homogeneous polynomial $P : \mathbb{R}^m \rightarrow \mathbb{R}$ is nondegenerate relative to a collection of subspaces $\{V_j\}$. Let the sets Y_r be defined as above, where the difference operators L_r satisfy (7.2). Then for any Lebesgue measurable functions f_j , any multi-parameter r , and almost every $x \in \mathbb{R}^m$,*

$$(7.6) \quad \text{If } x + y \in E_\varepsilon \text{ for every } y \in Y_r \text{ then at least one component of } r \text{ is } \leq C\varepsilon^{1/D}.$$

Proof. It suffices to prove this for locally integrable functions f_j , since the general case then follows from a limiting argument. Let $g = P - \sum_j f_j \circ \pi_j$. If $x + y \in E_\varepsilon$ for every $y \in Y_r$ then each term in the sum (7.4) is $O(\rho^{-D}\varepsilon)$, where ρ is the smallest component of r . Thus $1 = L_r g(x) = O(\rho^{-D}\varepsilon)$, so $\rho \leq C\varepsilon^{1/D}$. \square

We now abstract the situation to which the proof of Theorem 2.8 has been reduced. Let $Y \subset \mathbb{R}^m$ be an arbitrary finite, nonempty subset. Write $Y = \{y_\alpha : \alpha \in \mathcal{A}\}$ where \mathcal{A} is

a finite index set. To Y associate the sets $Y_r = \{\sum_{\alpha \in \mathcal{A}} r_\alpha \sigma_\alpha y_\alpha : \sigma \in \{0, 1\}^{|\mathcal{A}|}\}$, where $r = (r_\alpha)$, $\sigma = (\sigma_\alpha)$, each $r_\alpha \in \mathbb{R}^+$, and each $\sigma_\alpha \in \{0, 1\}$. Thus Y_r has cardinality $2^{|\mathcal{A}|}$.

Lemma 7.2. *Let Y be any nonempty finite subset of \mathbb{R}^m . Let $E \subset \mathbb{R}^m$ be measurable and contained in a fixed bounded set. Let $\varepsilon \in (0, 1]$. Suppose that for every $r \in (0, 1]^\mathcal{A}$, for almost every $x \in \mathbb{R}^m$, if $x + y \in E$ for every $y \in Y_r$ then at least one component r_α is $\leq \varepsilon$. Then $|E| \leq C\varepsilon^\delta$ where $C, \delta \in \mathbb{R}^+$, δ depends only on the cardinality of Y , and C depends only on Y .*

Proof. The special case where Y_r is simply the set of all vectors $\sum_{j=1}^m r_j \sigma_j e_j$, where e_j are the coordinate vectors, and $r = (r_1, \dots, r_m)$, was treated in [1]. The general case may be reduced to that special case by the following lifting argument.

Introduce $\mathbb{R}^M = \mathbb{R}_x^m \times \mathbb{R}_t^{|\mathcal{A}|}$, adding one real coordinate for each index α . Let $e_\alpha \in \mathbb{R}^{|\mathcal{A}|}$ be the unit vector corresponding to the α -th coordinate. Define $E^\dagger = E \times B$ where B is a fixed large ball in $\mathbb{R}^{|\mathcal{A}|}$.

Introduce the shear transformation $T : \mathbb{R}^M \rightarrow \mathbb{R}^M$ defined by $T(x, t) = (x - \sum_\alpha t_\alpha y_\alpha, t)$, where $t = \sum_\alpha t_\alpha e_\alpha$, and let $E^\ddagger = T(E^\dagger)$. Then for any r, σ, s and almost every x ,

$$x + \sum_\alpha r_\alpha \sigma_\alpha y_\alpha \in E \text{ if and only if } T(x, s) + (0, \sum_\alpha r_\alpha \sigma_\alpha e_\alpha) \in E^\ddagger.$$

Indeed, $x + \sum_\alpha r_\alpha \sigma_\alpha y_\alpha \in E$ is equivalent to $(x + \sum_\alpha r_\alpha \sigma_\alpha y_\alpha, s + \sum_\alpha r_\alpha \sigma_\alpha e_\alpha) \in E^\dagger$, since E^\dagger is invariant under translations in the second set of variables. Next

$$\begin{aligned} T(x + \sum_\alpha r_\alpha \sigma_\alpha y_\alpha, s + \sum_\alpha r_\alpha \sigma_\alpha e_\alpha) \\ &= (x + \sum_\alpha r_\alpha \sigma_\alpha y_\alpha - \sum_\alpha (s_\alpha + r_\alpha \sigma_\alpha) y_\alpha, s + \sum_\alpha r_\alpha \sigma_\alpha e_\alpha) \\ &= (x - \sum_\alpha s_\alpha y_\alpha, s + \sum_\alpha r_\alpha \sigma_\alpha e_\alpha) \\ &= T(x, s) + (0, \sum_\alpha r_\alpha \sigma_\alpha e_\alpha). \end{aligned}$$

Thus

$$\begin{aligned} x + \sum_\alpha r_\alpha \sigma_\alpha y_\alpha \in E &\Leftrightarrow (x + \sum_\alpha r_\alpha \sigma_\alpha y_\alpha, s + \sum_\alpha r_\alpha \sigma_\alpha e_\alpha) \in E^\dagger \\ &\Leftrightarrow T(x + \sum_\alpha r_\alpha \sigma_\alpha y_\alpha, s + \sum_\alpha r_\alpha \sigma_\alpha e_\alpha) \in E^\ddagger \\ &\Leftrightarrow T(x, s) + (0, \sum_\alpha r_\alpha \sigma_\alpha e_\alpha) \in E^\ddagger. \end{aligned}$$

Let $Y_r^\ddagger = \{(0, \sum_\alpha r_\alpha \sigma_\alpha e_\alpha)\} \subset \mathbb{R}^M$. Suppose now that whenever $x + y \in E$ for every $y \in Y_r$, some component r_α is $\leq \varepsilon$. Then for any $z \in \mathbb{R}^M$, if $z + y \in E^\ddagger$ for every $y \in Y_r^\ddagger$, then some $r_\alpha \leq \varepsilon$.

By Fubini's theorem, we may freeze the variable x , reducing matters to $\mathbb{R}^{|\mathcal{A}|}$; we are given (for almost every $x \in \mathbb{R}^m$) a set E^* , contained in a fixed bounded subset of $\mathbb{R}^{|\mathcal{A}|}$, such that for any $z \in \mathbb{R}^{|\mathcal{A}|}$, if $z + \sum_\alpha r_\alpha \sigma_\alpha e_\alpha \in E^*$ for every $\sigma \in \{0, 1\}^{|\mathcal{A}|}$ then some $r_\alpha \leq \varepsilon$. It was shown in [1] that this forces $|E^*| \leq C\varepsilon^\delta$ for some $\delta > 0$. See Lemma 3.7 of [3] for a more precise discrete analogue which implies the continuum version. \square

Proof of Theorem 2.8. Let $P, \{V_j\}, g_j, \varepsilon$ be as in Theorem 2.8, and let D be the degree of P .

In proving the theorem, we may suppose without loss of generality that the homogeneous part of P of degree D is nondegenerate with respect to $\{V_j\}$, for if not, it may be removed by replacing P by $P - \sum_j p_j \circ \pi_j$ for suitable polynomials p_j of degree D . Let the difference operators L_r and associated finite sets Y be associated to P_D as in Lemma 7.1 and the accompanying discussion.

Set $F = P - \sum_j g_j \circ \pi_j$ and consider its sublevel set $E_\varepsilon = \{z : |F(z)| < \varepsilon\}$. The difference operators L_r used in the proof of Lemma 7.1 satisfy $L_r(P_D - \sum_j g_j \circ \pi_j) \equiv 1$, but they annihilate all polynomials of degrees $< D$, so also satisfy $L_r(P - \sum_j g_j \circ \pi_j) \equiv 1$. Therefore the proof of Lemma 7.1 demonstrates that (disregarding a set of measure zero) if $x + y \in E_\varepsilon$ for all $y \in Y_r$ then the smallest component of r is $\lesssim \varepsilon^{1/D}$. It now suffices to invoke Lemma 7.2. \square

8. FURTHER RESULTS

Theorem 2.5 is a direct consequence of Lemma 3.6 together with Theorem 2.3. The following extension of Theorem 2.2 will be used to derive Theorem 2.4.

Theorem 8.1. *Let $\{V_j\}$ be any finite collection of subspaces of \mathbb{R}^m , and let $\{W_i\}$ be any finite collection of codimension one subspaces of \mathbb{R}^m . If $\{V_j\}$ has the uniform decay property, then so does $\{V_j\} \cup \{W_i\}$.*

Proof. It suffices to prove this in the case where a single codimension one subspace W is given. We will first prove that any nondegenerate polynomial has the decay property, then address the uniformity issue at the end of the proof. Choose coordinates with respect to which $W = \{(x', x_m) : x_m = 0\}$. Let a polynomial P be nondegenerate relative to the augmented collection $\{V_j\} \cup \{W\}$. We may assume that no subspaces V_j are contained in W , for any such spaces may be deleted from $\{V_j\}$ without affecting the nondegeneracy of P .

$\partial P / \partial x_m$ is nondegenerate relative to $\{V_j\}$. For if not, then there exists a polynomial decomposition $\partial P / \partial x_m = \sum_j q_j \circ \pi_j$. Since no space V_j is contained in W , $\partial(Q_j \circ \pi_j) / \partial x_m = (v_j \cdot \nabla Q_j) \circ \pi_j$ for certain nonzero vectors v_j . Therefore there exist polynomials Q_j such that $\partial(Q_j \circ \pi_j) / \partial x_m = q_j \circ \pi_j$, and hence by setting $\tilde{P} = \sum_j Q_j \circ \pi_j$ we have $\partial(P - \tilde{P}) / \partial x_m \equiv 0$. Thus $P - \sum_j Q_j \circ \pi_j$ is a function of x_m alone, whence P is degenerate.

Let d be the degree of P . It now follows that there exists $z \in \mathbb{R}^m$ for which $P_z(x) = P(x', x_m) - P(x', x_m + z)$ is nondegenerate relative to $\{V_j\}$. If we consider the quotient space \mathcal{P} of all polynomials of degrees $\leq d$ modulo the subspace of all such polynomials which are degenerate relative to $\{V_j\}$ with an inner product structure, then $\|P_z\|_{\mathcal{P}}^2$ is a polynomial in z which does not vanish identically. Hence there exist $C, \delta \in \mathbb{R}^+$ such that for any ball B of fixed finite radius, for any $\varepsilon > 0$,

$$(8.1) \quad |\{z \in B : \|P_z\|_{\mathcal{P}}^2 < \varepsilon\}| \leq C\varepsilon^\delta.$$

We may now argue as in the proof of Theorem 2.1 to conclude the proof that P has the decay property.

The same reasoning as above demonstrates that if a family $\{P_\alpha\}$ of polynomials of uniformly bounded degrees is uniformly nondegenerate relative to $\{V_j\}$, then $\{\partial P_\alpha / \partial x_m\}$ is uniformly nondegenerate relative to $\{V_j\} \cup \{W\}$. Hence $\|\partial P_\alpha / \partial x_m\|_{\mathcal{P}}$ is bounded away

from zero, uniformly in α . From this the sublevel set bound (8.1) follows by elementary reasoning. \square

Corollary 8.2. *Let a real-valued polynomial P be nondegenerate relative to a finite collection $\{V_j\}$ of subspaces of \mathbb{R}^m , and suppose that V_j has codimension one for all $j > 1$. Then $\{V_j\}$ has the uniform decay property. Moreover*

$$(8.2) \quad |\Lambda_\lambda(f_1, \dots, f_n)| \leq C|\lambda|^{-\varepsilon} \|f_1\|_2 \prod_{j>1} \|f_j\|_\infty,$$

with uniform bounds if P belongs to a family of uniformly nondegenerate polynomials.

Indeed, this follows from the L^∞ bound by interpolating with the trivial bound $\lesssim \|f_1\|_1 \prod_{j>1} \|f_j\|_\infty$.

Proof of Theorem 2.4. We will prove power decay in the stronger form (8.2). The case $n = 1$ is a well-known fact, as discussed earlier. Let W be the span of those V_j with $2 \leq j \leq M+1$, and \tilde{W} be the span of those with $M+1 < j \leq n$, $n \leq 1+2M$ and $(1+M)\kappa \leq m$. By the general position hypothesis, W has dimension $M\kappa$ and \tilde{W} has dimension $(n-M-1)\kappa$; both of these dimensions are $\leq m-1$.

Let functions $f_j \in L^2(V_j)$ be given. As usual, we may assume each f_j to be supported in a fixed bounded subset of V_j .

The proof is divided into two main cases, depending on whether or not f_1 is λ -uniform, as defined in Definition 5.1. Let $A(\lambda)$ be the best constant in the inequality (8.2). If f_1 is not λ -uniform, then it follows by induction on n , as in the proof of Theorem 2.1, that $A(\lambda) \leq C|\lambda|^{-\delta} + (1-|\lambda|^{-\tau})A(\lambda)$.

Define functions F, \tilde{F} on W, \tilde{W} respectively by the relations $F \circ \pi = \prod_{j=2}^{M+1} f_j \circ \pi_j$, $\tilde{F} \circ \tilde{\pi} = \prod_{j>M+1} f_j \circ \pi_j$ where $\pi, \tilde{\pi}$ denote the orthogonal projections from \mathbb{R}^m onto W, \tilde{W} respectively. Then by general position, $\|F\|_2 \lesssim \prod_{j=2}^{M+1} \|f_j\|_2$ and $\|\tilde{F}\|_2 \lesssim \prod_{j>M+1} \|f_j\|_2$.

Consider next the case where f_1 is λ -uniform. In this case the hypothesis that P is nondegenerate plays no role. Our integral is

$$\Lambda_\lambda = \int_{\mathbb{R}^m} e^{i\lambda P(x)} (F \circ \pi)(\tilde{F} \circ \tilde{\pi})(f_1 \circ \pi_1) \eta.$$

Now P may be nondegenerate relative to the collection of three subspaces V_1, W, \tilde{W} . If it is, then the proof is complete by virtue of the preceding corollary, in its more precise version (8.2). Thus we may assume henceforth that P is decomposable as $P = p \circ \pi + \tilde{p} \circ \tilde{\pi} + q \circ \pi_1$ for certain polynomials p, \tilde{p}, q .

Set $G = e^{i\lambda p} F$, $\tilde{G} = e^{i\lambda \tilde{p}} \tilde{F}$, and $g = e^{i\lambda q} f_1$. Our integral becomes

$$\int (G \circ \pi)(\tilde{G} \circ \tilde{\pi})(g \circ \pi_1) \eta = c \iiint \widehat{G}(\xi) \widehat{\tilde{G}}(\tilde{\xi}) \widehat{g}(\zeta) \widehat{\eta}(\pi^* \xi + \tilde{\pi}^* \tilde{\xi} + \pi_1^* \zeta) d\xi d\tilde{\xi} d\zeta$$

where π_1^* is the adjoint of $\pi_1 : \mathbb{R}^m \rightarrow V_1$, and analogously for $\pi^*, \tilde{\pi}^*$. The λ -uniformity condition gives a bound $\|\widehat{g}\|_{L^\infty} = O(|\lambda|^{-\tau})$ for some fixed $\tau > 0$. Therefore for $|\lambda| \geq 1$

$$|\Lambda_\lambda| \lesssim |\lambda|^{-\tau} \|f_1\|_2 \iiint |\widehat{G}(\xi) \widehat{\tilde{G}}(\tilde{\xi})| \cdot |\widehat{\eta}(\pi^* \xi + \tilde{\pi}^* \tilde{\xi} + \pi_1^* \zeta)| d\xi d\tilde{\xi} d\zeta.$$

It therefore suffices to have

$$(8.3) \quad \sup_{\tilde{\xi}} \int |\widehat{\eta}(\pi^* \xi + \tilde{\pi}^* \tilde{\xi} + \pi_1^* \zeta)| d\xi d\zeta < \infty$$

and the same with the roles of $\tilde{\xi}, \xi$ interchanged. The mapping $\mathbb{R}^{\kappa+M\kappa} \ni (\zeta, \xi) \rightarrow \pi^*\xi + \pi_1^*\zeta \in \mathbb{R}^m$ is linear and injective by the general position hypothesis, since $(1+M)\kappa \leq m$, and η may be taken to belong to C^K for any preassigned K , so that $\hat{\eta}$ decays rapidly. Thus (8.3) holds. Since the roles of $\xi, \tilde{\xi}$ are symmetric, it continues to hold when they are interchanged. \square

More general results, in which the dimensions of the subspaces V_j are not all required to be equal, can be proved in the same way.

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